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2001 J. Phys. A: Math. Gen. 34 8333

(http://iopscience.iop.org/0305-4470/34/40/310)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 8333–8343 PII: S0305-4470(01)27629-X

The identification of Young tableaux with angular momentum states

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Received 25 July 2001
Published 28 September 2001
Online at stacks.iop.org/JPhysA/34/8333

Abstract

Young tableaux are used to label the basis vectors of the standard or Young–Yamanouchi basis of the symmetric group. Despite being used for this purpose for some time, a physical interpretation of what they mean has not been given. Weyl tableaux however, which label the basis vectors of the standard or Gelfand basis of the unitary group, do have a physical interpretation. Weyl tableaux correspond to antisymmetrized states with definite total spin, definite spin projection and definite total angular momentum projection. We discuss how a previously well established link between Young and Weyl tableaux may imply Young tableaux are similarly associated with such states. If such an association could be made, calculations using symmetric group bases could be reduced to simple angular momentum manipulations. In particular this would greatly increase the efficiency of calculating the coefficients of fractional parentage of S_n, which can in turn be used to calculate unitary recoupling coefficients. We present a methodology for determining whether such an association exists.

PACS number: 02.20.-a

1. Introduction

Unitary groups can be used to provide canonical labels for states of physical systems, whether the system be atomic, nuclear or particle. Using the standard bases, $U_N \supset U_{N-1} \supset \cdots \supset U_2 \supset U_1$, no multiplicity labels are required, whereas in general they are. For example, the groups in Racah's original (Racah 1943, 1949) seniority schemes for atomic spectroscopy do not completely specify the states. It is useful that the Weyl tableaux which represent, or simply are, the basis functions of the standard basis have a physical interpretation in terms of angular momentum basis functions (Drake *et al* 1975).

There are similar standard bases for the symmetric groups, $S_n \supset S_{n-1} \supset \cdots \supset S_2 \supset S_1$, in which there is again no need for multiplicity. Although the symmetric groups are themselves more fundamental than the unitary groups, their basis functions, labelled by the Young tableaux,

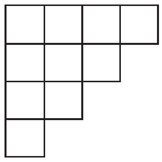

Figure 1. The Ferrers diagram associated with the irrep [4 3 2 1] of S_{10}.

have to date been given no physical interpretation in terms of conserved properties such as angular momentum. Such an interpretation would simplify calculations within, and related to, the standard symmetric bases. It would make the calculation of symmetric group coefficients of fractional parentage (CFPs) simpler. Since CFPs of symmetric groups can be used to calculate unitary group recoupling coefficients, the unitary group calculations will also be simplified.

The question we would like to answer then is, 'Do the Young tableaux have a fundamental, consistent physical interpretation?'. It is known from Schur–Weyl duality that the unitary and symmetric groups are related. Furthermore it is known that Weyl tableaux can be interpreted in terms of angular momentum states. Therefore it would be useful to find an explicit link between the Young tableaux and Weyl tableaux. We demonstrate that such a link exists, although the relationship is between a single Young tableau and an infinite set of Weyl tableaux. Clearly such a relationship is unhelpful; it is simply the well known duality. We therefore propose a method of searching for a restricted set of Weyl tableaux in the relationship, if possible to give a one-to-one correspondence. Our methodology involves searching for an appropriate restriction using a physical calculation, namely CFPs, as a guideline.

2. Background and notation

A partition $[\lambda]$ of the number n into i parts may be written as $[\lambda_1, \lambda_2, \ldots, \lambda_i]$ such that $\sum_{j=1}^{i} \lambda_j = n$ and so the λ_j are weakly decreasing ($\lambda_j \geqslant \lambda_{j+1}, \forall j$).

A useful way to manipulate the irreps through the partition labels is to use diagrams and tableaux. By forming a left-justified array with λ_j boxes on the jth row with the kth row below the $(k-1)$th row, we obtain a Ferrers or Young diagram. For example in figure 1 we give the Ferrers diagram associated with the irrep [4 3 2 1] of S_{10}.

Now we want to examine the basis vectors which span a given irrep. We shall consider two different bases, the standard or Young–Yamanouchi basis and the split basis. In the standard, one labels the basis vectors or eigenstates by Young tableaux, different fillings of the Ferrers diagram with the numbers 1 to n increasing across rows and down columns. The number of Young tableaux is equal to the dimension of the irrep and we enumerate the basis vectors of an irrep ν by an integer m running from 1 to the dimension of ν. Loosely the Young tableaux describe the symmetry relations between the $(i-1)$-particle state and the ith particle for all $2 \leqslant i \leqslant n$. They are equivalent to labelling the basis vectors by irreps (Ferrers diagrams) for each $S_i, i \leqslant 1 \leqslant n$. It is not known how to physically interpret the standard basis labels however, in the sense that the tableaux are not known to specifically relate to any observables or conserved quantities, such as orbital or spin angular momentum. The particular advantage of the basis is that there are no multiplicity labels; that is, each Young tableau is distinct. A typical Young tableau for [4 3 2 1] of S_{10} is given in figure 2.

The split basis was first introduced by Elliott *et al* (1953). The basis vectors are labelled by the irrep of S_n, by Young tableaux of S_a and S_b where $n = a + b$ and by a product multiplicity τ arising in the generation of the S_n irrep from the S_a and S_b irreps.

1	3	4	7
2	5	10	
6	9		
8			

Figure 2. A typical Young tableau for [4 3 2 1] of S_{10}.

3. Weyl tableaux in terms of angular momentum

The Weyl tableaux are similar to the Young tableaux; they are associated with partitions. Weyl tableaux are used to label basis vectors in the standard basis of the unitary group[1], $U_n \supset U_{n-1} \supset \cdots \supset U_2 \supset U_1$. The irreps of U_n are labelled by Ferrers diagrams, however not just those associated with partitions of n. Rather the irreps of U_n are associated with integer partitions of n parts. The basis vectors are then Weyl tableaux, which are obtained from the Ferrers diagrams by filling in with the numbers 1 to n. In the diagram the numbers must strictly increase down columns and weakly increase across rows, thus numbers can be repeated and used as often as otherwise allowable.

Weyl tableaux have a definite meaning in terms of angular momentum[2] (Drake *et al* 1975). Each corresponds to a unique antisymmetrized n-particle state with definite total spin S, definite spin projection S_z and definite component of total angular momentum in the z direction J_z. They are however mixtures of the states with all possible values of the total angular momentum J.

An example is discussed in Drake *et al* (1975) for the Weyl tableaux associated with 'the doublet states of f^3'. The f implies we are dealing with orbital angular momentum $l = 3$ for each particle. Thus N, the maximum of the orbital angular momentum for each particle, is $2l + 1 = 7$. The doublet states imply definite total spin $S = \frac{1}{2}$.

4. Relating Young and Weyl tableaux

In this section we shall construct vector spaces, associated with unitary and symmetric groups, which are spanned by the set of particle states. The product state of single-particle states defines a possible state of the entire system. We follow Kaplan (1975) and more particularly Elliott and Dawber (1979) in setting up the appropriate vector space, which we use to link the basis functions of the symmetric and unitary groups.

In the words of Elliott and Dawber (1979, p 461) we emphasize that 'the device of writing state labels into the Young diagrams for $U(N)$ is quite distinct from the device of writing in particle labels ... for S_n'. It is important to realize that the identification we initially make relies on the underlying functions ϕ_i and Φ. This identification is however infinite, hence the statement in quotes above. The methodology we present in section 5.1 provides a means for making a more localized connection between Weyl and Young tableaux based on physical calculations.

Consider n particles which may each be put into any of p single-particle states ϕ_i so that $\phi_i(k)$ signifies that particle k is in state i.

We associate with each partition of n, $\lambda = [n_1 n_2 \ldots]$, a product function

$$\Phi([n_1 n_2 \ldots]) = \phi_1(1)\phi_1(2)\ldots\phi_1(n_1)\phi_2(n_1+1)\phi_2(n_1+2)\ldots \\ \ldots\phi_2(n_1+n_2)\phi_3(n_1+n_2+1) \qquad (4.1)$$

[1] We use the lower-case n rather than upper-case N for consistency with Drake *et al* (1975) in the maximum orbital angular momentum for each particle.

[2] We refer to the Young tableaux of Drake *et al* (1975) more precisely as Weyl tableaux.

for which the first n_1 particles are in the ϕ_1 state, the next n_2 particles are in state ϕ_2 and so on. This can be written in a concise but still explicit way as

$$\Phi([n_1 n_2 \ldots n_p]) = \Pi_{j=1}^{p} \Pi_{i=1}^{n_j} \left(\phi_j \left(i + \sum_{k=1}^{j-1} n_j \right) \right). \tag{4.2}$$

A more concise notation still is

$$\Phi([\lambda]) = |\{i\}\rangle \tag{4.3}$$

where $\{i\}$ represents the ordered list of states the ordered list of particles appear in. For example, $|ijk\rangle = \phi_i(1)\phi_j(2)\phi_k(3)$. We shall later subscript an S on the Φ to distinguish between the symmetric and unitary cases.

For each partition consider the set of independent product functions $P\Phi$, where P is any permutation of the particles. Although there are n particles the indistinguishability of the particles in the same states means there are

$$\frac{n!}{\Pi_i n_i!} \tag{4.4}$$

independent functions. This set of states defines or spans a vector space, $L(\lambda)$, which is invariant under permutations and therefore provides a representation of S_n. Although this representation is in general reducible it has the distinct advantage of having characters which can be calculated easily. When $L(\lambda)$ is reduced the basis functions of the irreps in the decomposition can be expressed in terms of the $P\Phi$ functions in L (see, for example, section 17.9 of Elliott and Dawber). We can represent the irreps in the standard basis, taking appropriate combinations of the $P\Phi$ so that the basis functions are Young tableaux. This process is best illustrated initially through some simple examples.

For any n the one-dimensional representations $[n]$ and $[1^n]$ correspond to the totally symmetric, or identity, and the antisymmetric, or alternating, representations respectively. The first irrep not of this kind is [2, 1] of S_3 for which

$$\Phi([2, 1]) = \phi_1(1)\phi_1(2)\phi_2(3) = |112\rangle. \tag{4.5}$$

The space $L([2, 1])$ has the basis vectors $|112\rangle$, $|121\rangle$ and $|211\rangle$. Using character tables we find our representation reduces to the sum of two irreps, [3] and [2, 1]. The totally symmetric, and normalized, vector is

$$\theta([3]) = \frac{1}{\sqrt{3}} \{|112\rangle + |121\rangle + |211\rangle\}. \tag{4.6}$$

Then the other pair of basis vectors is chosen to be orthogonal, normalized and irreps of S_2 on the first two particles:

$$\theta([2, 1]a) = \frac{1}{\sqrt{6}} \{2|112\rangle - |121\rangle - |211\rangle\} \tag{4.7}$$

$$\theta([2, 1]b) = \frac{1}{\sqrt{2}} \{|121\rangle - |211\rangle\rangle\}. \tag{4.8}$$

The first of those two functions is symmetric under the exchange of the first two particles, so is associated with the Young tableaux[3] $\begin{matrix} 1 & 2 \\ 3 & \end{matrix}$. The second of the functions is antisymmetric under the exchange of the first two particles, so is associated with the Young tableaux $\begin{matrix} 1 & 3 \\ 2 & \end{matrix}$.

[3] Generally we shall omit the boxes and just give the numbers in the appropriate shape for small tableaux such as these.

Now let us turn our attention to the unitary group $U(N)$. We construct a product space defined by the set of N^n products, where n is the number of particles and N the number of states,

$$\Phi_U = \phi_i(1)\phi_j(2)\ldots\phi_p(n) = |ij\ldots p\rangle\,. \tag{4.9}$$

The suffix again labels the state that each particle is in and is known as the state label. The label in brackets is again the particle label. The ϕ are basis vectors in an N-dimensional space. It is appropriate to say the ϕ are single-particle states if the N-dimensional space is taken to be the eigenspace of single-particle states. Then if P is the set of all permutations of the particles, the spaces spanned by $P\Phi_S$ and by $P\Phi_U$ are the same if the number of particles and states are the same. We can thus associate basis functions of the symmetric group, Young tableaux, with basis functions of the unitary group, Weyl tableaux. Before making this identification it is useful to look at some examples of the decomposition of the unitary product space into Weyl tableaux.

The space $P\Phi_U$ decomposes into a sum over all irreps of $U(N)$ with the sum of the partition labels equalling n; however, some of the irreps may occur more than once.

4.1. Examples

- $N = 2$, $n = 2$, four product functions:

$$|11\rangle, |12\rangle, |21\rangle, |22\rangle. \tag{4.10}$$

Subspace basis vectors:

$$\begin{array}{r|lll} L([2]) & |11\rangle & \frac{1}{\sqrt{2}}(|12\rangle + |21\rangle) & |22\rangle \\ L([1,1]) & & \frac{1}{\sqrt{2}}(|12\rangle - |21\rangle)\,. & \end{array} \tag{4.11}$$

Reading left to right, top to bottom those correspond to the Weyl tableaux 1 1, 1 2, 2 2 and $\begin{smallmatrix}1\\2\end{smallmatrix}$.

- $N = 3$, $n = 2$, nine product functions. Subspace basis vectors:

$$\begin{array}{r|lll} L([2]) & |11\rangle, |22\rangle, |33\rangle & \frac{1}{\sqrt{2}}(|12\rangle + |21\rangle) & \frac{1}{\sqrt{2}}(|13\rangle + |31\rangle) \\ & & \frac{1}{\sqrt{2}}(|23\rangle + |32\rangle) & \\ L([1,1]) & & \frac{1}{\sqrt{2}}(|12\rangle - |21\rangle) & \frac{1}{\sqrt{2}}(|13\rangle - |31\rangle) \\ & & \frac{1}{\sqrt{2}}(|23\rangle - |32\rangle)\,. & \end{array} \tag{4.12}$$

Reading left to right, top to bottom again those correspond to the Weyl tableaux 1 1, 2 2, 3 3, 1 2, 1 3, 2 3 of $L([2])$ and $\begin{smallmatrix}1\\2\end{smallmatrix}, \begin{smallmatrix}1\\3\end{smallmatrix}, \begin{smallmatrix}2\\3\end{smallmatrix}$ of $L([1, 1])$.

- $N = 2$, $n = 3$, eight product functions:

$$|111\rangle, |112\rangle, 121\rangle, |122\rangle, |211\rangle, |212\rangle, |221\rangle, |222\rangle. \tag{4.13}$$

Subspace basis vectors:

$$\begin{array}{r|ll} L([3]) & |111\rangle & \frac{1}{\sqrt{3}}(|112\rangle + |121\rangle + |211\rangle) \\ & |222\rangle & \frac{1}{\sqrt{3}}(|221\rangle + |212\rangle + |122\rangle) \\ L([2,1]) & \frac{1}{\sqrt{6}}(2|112\rangle - |121\rangle - |211\rangle) & \frac{1}{\sqrt{6}}(2|221\rangle - |212\rangle - |122\rangle) \\ & \frac{1}{\sqrt{2}}(|121\rangle - |211\rangle) & \frac{1}{\sqrt{2}}(|212\rangle - |122\rangle)\,. \end{array} \tag{4.14}$$

The identification of these basis functions with Weyl tableaux proceeds somewhat differently from the symmetric group situation (Elliott and Dawber 1979). The results for $L([2, 1])$, for example, are $\begin{smallmatrix}1 & 1\\2 & \end{smallmatrix}, \begin{smallmatrix}1 & 2\\2 & \end{smallmatrix}, \begin{smallmatrix}1 & 1\\2 & \end{smallmatrix}, \begin{smallmatrix}1 & 2\\2 & \end{smallmatrix}$.

This means that there are two copies of [2 1] in $L([2\ 1])$.

5. Matching Weyl tableaux to Young tableaux

We showed in section 4.1 that we can represent the Weyl tableaux in terms of single-particle basis state functions ϕ. Similarly we showed in section 4 that we can represent the Young tableaux in terms of single-particle basis state functions, ϕ. Since the ϕ are the same we can therefore equate or at least relate the Weyl tableaux and the Young tableaux. However the identification is such that provided the unitary group is large enough to contain the appropriate irrep there will be another Weyl tableau to match a Young tableau. Thus in some sense there are an infinite number of Weyl tableaux associated with a single Young tableau. But how should we restrict this relationship, since, while each partition is used once for the symmetric groups, it is used in all unitary groups above a certain partition-dependent size?

Although we cannot state this relation, at this time, we propose a scheme in the next section for determining the relation, or at the very least for testing whether it is possible to do so.

5.1. The searching scheme

Our method of searching is based around the calculation of CFPs, which can be calculated in various ways. Essentially one checks to see what angular momentum states we can replace the Young tableaux with and still get the correct CFP. There are several places where one can make choices. By repeating the steps with different choices we aim to find an angular momentum association that gives the correct CFP. The method is sketched in figure 3. The full methodology is presented below and includes discussion as necessary.

- Take a CFP. It is specified in terms of two split basis tableaux $\lambda_1 m_1$ and $\lambda_2 m_2$, and a standard basis tableaux λm. We begin the procedure by considering the two simplest CFPs, those associated with the coupling of two lots of [1] of S_1 to give either [2] or [1^2] of S_2. The choice of the angular momentum state associated with [1] is the first fundamental variant in this procedure.
- Convert the $\lambda_1 m_1$ and $\lambda_2 m_2$ to $\Phi_1 = \Phi(\lambda_1)$ and $\Phi_2 = \Phi(\phi_2)$ as in section 4.
- Choose Φ' by one of the following two methods.
  (1) By adjacency of $\Phi_1\Phi_2$ in their explicit functional form. Thus, for example, if $\Phi_1 = \phi_1(1)\ \phi_1(2)\phi_2(3)$ and $\Phi_2 = \phi_1(4)\ \phi_1(5)\phi_2(6)$, then $\Phi' = \phi_1(1)\phi_1(2)\ \phi_2(3)\phi_1(4)\phi_1(5)\phi_2(6)$ or after reordering $\Phi' = \phi_1(1)\phi_1(2)\ \phi_1(4)\phi_1(5)\phi_2(3)\phi_2(6)$.
  (2) Use *jeu de taquin* (Schützenberger 1963, McAven *et al* 1998) backwards to obtain from the pair of tableaux a standard tableaux $\lambda m'$, which we then convert to Φ. Where multiplicities occur we may either take the simpler option of matching the first multiplicity to the first tableaux (in first-letter order) giving rise to the pair, or the more complicated option of taking more general orthogonal combinations. For the above case the two $\lambda m'$ are

$$\begin{matrix} 1 & 2 & 5 \\ 3 & 4 & \\ 6 & & \end{matrix} \tag{5.1}$$

  and

$$\begin{matrix} 1 & 2 & 5 \\ 3 & 6 & \\ 4 & & \end{matrix} \quad . \tag{5.2}$$

- By the functional correspondence between the Young and Weyl tableaux through the Φ (and ϕ_i), discussed in sections 4 and 4.1, treat Φ' as a Weyl tableaux.

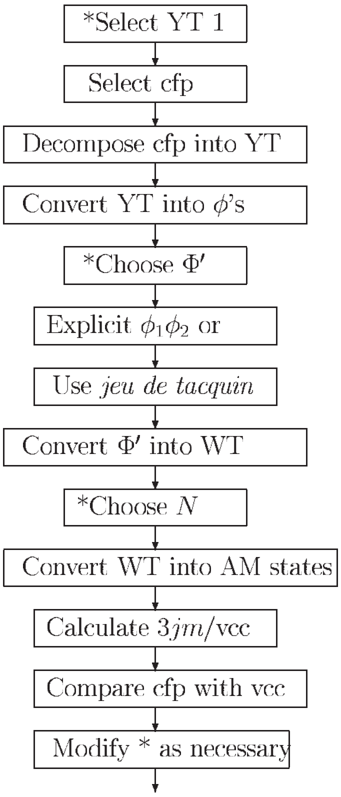

Figure 3. Our search method for obtaining a definitive physical link between Young and Weyl tableaux. Each box labelled with a * contains a choice which can be modified at the step *Modify * as necessary*. Those changes will be modified in order to search for the desired relationship. We use YT, WT and AM to represent Young tableaux, Weyl tableaux and angular momentum respectively.

- The correspondence above holds for all $N \geqslant n$. So we choose a single N value (initially) with the aim of testing to see whether the physical conditions imposed by the CFP requested restrict the correspondence.
- Having chosen N we follow Drake *et al* (1975) in converting the Weyl tableaux to angular momentum states, as discussed in section 3.
- From the angular momentum states we calculate the $3jm$ or vector coupling coefficients between the angular momentum state associated with Φ' and the angular momentum states associated with the simpler Φ_1 and Φ_2 states.
- Comparing the result of the last step with the CFP will give either a negative response, requiring a modification in N, or a change in the correspondence method. Success will be recorded and the next CFP can be considered. It may be possible that several values of N will give the desired CFP $= 3jm$; the meaning of such a situation would need to be considered.

This recursive method builds up appropriate angular momentum associations for the basis vectors of a larger symmetric group from a selected, consistent angular momentum association for the primitive basis vector (and irrep) [1].

6. A trial run of the search

Having presented the method we propose for identifying an appropriate relationship between the Young tableaux and angular momentum states, the next step is to consider some simple examples. This is particularly important in attempting to identify the primitive basis vector [1] with angular momentum vectors, and in order to provide some initial guidance in making selections at the critical choice parts of the search. This section will also serve to clarify the method.

The two simplest CFPs are those coupling the primitive irrep [1] with itself to give either the symmetric irrep [2] or the anti-symmetric irrep $[1^2]$. In the notation of Chen *et al* (1983) those CFPs are written as

$$\left\langle \begin{matrix}[2]\\1\end{matrix}\right| [2], \begin{matrix}[1]\\1\end{matrix} \quad \begin{matrix}[1]\\1\end{matrix}\left.\right\rangle \qquad \left\langle \begin{matrix}[1^2]\\1\end{matrix}\right| [1^2], \begin{matrix}[1]\\1\end{matrix} \quad \begin{matrix}[1]\\1\end{matrix}\left.\right\rangle . \tag{6.1}$$

In this notation a general CFP is written in the form

$$\left\langle \begin{matrix}\lambda\\m\end{matrix}\right| \lambda, \begin{matrix}\lambda_1\\m_1\end{matrix} \quad \begin{matrix}\lambda_2\\m_2\end{matrix}\left.\right\rangle \tag{6.2}$$

where m, m_1 and m_2 are multiplicity labels associated with the respective irreps.

Those CFPs are both equal trivially to unity since they do not describe a basis transformation at all. Each side of the bracket is adapted to the Young–Yamanouchi basis. The 'method of search' is not trivial in its application to them however. The first choice, of primitive, we leave until the end of the process. We will then *derive* the primitive by insisting that the CFP and $3jm$ values be equal.

The second choice is in the method proper and requires obtaining Φ'. This example indicates that if one chooses the first of the two methods, explicit $\phi_1\phi_2$ adjacency, then the distinction at the angular momentum end of the search will need to be built in through a less simple choice of N. Thus N will be a function of the shape of $[\lambda]$, rather than just the number of boxes in $[\lambda]$.

For each case it is therefore natural to try to identify Φ' as the irrep and its multiplicity, that is the only Young tableau associated with that irrep. The last factor we can specify as the same for each case is to try N as 2.

For $\lambda = [1^2]$ in $U(2)$ the irrep $[1^2]$ is one dimensional:

$$\boxed{\begin{matrix}1\\2\end{matrix}} \;. \tag{6.3}$$

The total number of boxes $n = 2$ and the angular momentum associated with each Weyl tableau box is $l = \frac{1}{2}$. The total spin $S = 1$ is given by counting $2S$, the number of horizontally unpaired boxes. The projection of angular momentum is given by $n(l + 1)$ less the sum of the entries in the tableaux, thus it is $M_L = 0$. Given the presence of only one Weyl tableau associated with the irrep the total angular momentum is $L = 0$. The projection of the spin, $M_S = 0$.

For $\lambda = [2]$ in $U(2)$ the irrep [2] is three dimensional:

$$\boxed{1\,|\,1} \qquad \boxed{1\,|\,2} \qquad \boxed{2\,|\,2} \;. \tag{6.4}$$

The total number of boxes $n = 2$ and the angular momentum associated with each Weyl tableau box is $l = \frac{1}{2}$. The total spin $S = 0$ is given by counting $2S$, the number of horizontally unpaired boxes. Clearly then the spin projection $M_S = 0$. The projection of angular momentum is given by $n(l + 1)$ less the sum of the entries in the tableaux, thus it is 1, 0 and -1 for the respective tableaux above. Given the three Weyl tableaux the angular momentum is necessarily $L = 1$.

In summary then

	S	M_S	L	M_L
$[1^2]$	1	0	0	0
$[2]$	0	0	1	0

(6.5)

Two $3jm$ equations can be generated using

$$1 = \begin{pmatrix} L_1 & L_2 & L \\ M_{L_1} & M_{L_2} & M_L \end{pmatrix} \begin{pmatrix} S_1 & S_2 & S \\ M_{S_1} & M_{S_2} & M_S \end{pmatrix} \tag{6.6}$$

with the appropriate values for the two cases. Here the two equations generated by equation (6.6) are the same; only the order in which terms appear differs.

At this point we raise the question of whether we should make the CFP identification with $3jm$ or with vector coupling coefficients. The relationship between a $3jm$ and the associated vector coupling coefficient (VCC) is given by

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda \\ l_1 & l_2 & l \end{pmatrix}_r = |\lambda|^{-\frac{1}{2}} \begin{pmatrix} \lambda^* \\ l^* \end{pmatrix} \langle r\lambda^* l^* | \lambda_1 l_1, \lambda_2 l_2 \rangle \tag{6.7}$$

where $|\lambda|$ is the dimension of the irrep λ and $*$ is used to denote complex conjugation. The subscript r records product multiplicities. In the examples we are considering one of the λ values is unity so that one of the $\lambda^{-\frac{1}{2}}$ is $\frac{1}{\sqrt{3}}$ in each case. Thus it matters whether or not we choose to make the equality between CFP and $3jm$ or between CFP and VCC. Indeed we should be able to determine which is the simpler to relate to.

In this case however the point is mute, since the $3jm$ equations, in (6.6), cannot be satisfied, even by defining the tableaux 1 to be a multi-label state. This suggests that we have incorrectly chosen N for the two-box irreps. Note however that if each term in the expansion were to be pre-multiplied by $(-1)^{M_S+M_L}$ the result would be correct for VCCs. This would imply that the single-box tableau does not follow angular momentum rules, so this is not helpful. If it were possible to change the phase of the $3jm$

$$\begin{pmatrix} L_1 & L_2 & L \\ M_1 & M_2 & M \end{pmatrix} \tag{6.8}$$

by $(-1)^{M_2}$ the equation would be satisfied, by the basis state we shall introduce below.

Such a phase change not being possible we need to return to our search and take a different pathway. Let $f(L, M_L, S, M_S) = |(L\ M_L), (S\ M_S)\rangle$ represent a basis vector in the spin–orbital angular momentum basis and let $a = \frac{1}{2}$. We now propose that the one-box tableaux be associated with the normalized state

$$\tfrac{1}{2}\big[f(a, a, a, a) + f(a, a, a, -a) + f(a, -a, a, a) + f(a, -a, a, -a)\big]. \tag{6.9}$$

This seems a reasonable choice since it contains the primitive irreps and irrep-subirreps of the appropriate groups and chains. Having made this choice we can now use equation (6.6) in the reverse direction to previously. That is, knowing $L_1 = L_2$, M_{L_1}, M_{L_2}, $S_1 = S_2$, M_{S_1} and M_{S_2} we shall attempt to derive L, M_L, S and M_S for each of the two box cases.

This appears to be a non-trivial exercise since there are at least 16 terms, each the product of two $3jm$ or VCCs. There may well be more, since it must be remembered that the Weyl tableaux do not have definite L values associated with them but rather are linear combinations of them. If we find a suitable set of L, M_L, S and M_S for our Φ' we need to then find an N value which matches them up.

The problem is greatly reduced however by the sum rule on the SO_2 irreps in the $3jm$ and the pre-knowledge of S, M_S and M_L independent of the choice of N. Indeed with the symmetry of the $3jm$ we again find that the right-hand side of equation (6.6) goes to zero. Thus we see that this choice for the primitive one-box tableaux is not correct. We must return to our search and make alternative choices.

7. Summary

There is a well accepted duality between symmetric and unitary groups. We exploit this to obtain a relationship between the Young tableaux and Weyl tableaux. Those tableaux label the basis vectors of the symmetric and unitary group respectively. The initial correspondence relates a single Young tableaux to an infinite number of Weyl tableaux. We present a methodology for investigating the restriction to a finite, particular singular, set of Weyl tableaux based on the physical system in use.

Our method of search uses the CFP, which can be calculated in a relatively straightforward manner, to test possible restricted relationships between Weyl and Young tableaux in a structured manner. The method relies on making choices at several points. Those choices decide which relationship we are trying to confirm.

The aim of obtaining a correspondence is to provide a physical interpretation to the Young tableaux, even as the Weyl tableaux have an interpretation in terms of angular momentum states (Drake *et al* 1975). It may be that no such interpretation exists, it may be that angular momentum is not the correct conserved property to consider, but this is a step towards clarifying the uncertainty surrounding whether or not Young tableaux have a physical interpretation. Naturally a finite correspondence between Young and Weyl tableaux will provide a similar, although probably more complicated description of the Young tableaux. Such a representation of Young tableaux will allow for them to be more easily manipulated in calculations. There are several possible directions in which we might further investigate the results of any relationship that is found.

- Clearly the way the CFPs are used implies the calculation of CFPs will be more efficient and more physical.
- Algebraic expressions for some classes of S_n CFPs have previously been given (Kaplan 1961, McAven *et al* 1998), and can be generated for other cases using the approach of McAven *et al* (1998). Although it may not be possible to give a general expression we may be able to unify the expressions given to date or give expressions for other classes.
- More likely than obtaining the above is the possibility of obtaining a combinatorial recipe for deriving the CFPs, thus for ‘multiplying basis functions of the symmetric group’. A combinatorial recipe is weaker than an algebraic formula, but always works. The Littlewood–Richardson rule is a combinatorial recipe describing the multiplication of irreps of the symmetric group, and it may be possible to obtain an analogous rule. A selection rule for the product of basis tableaux has already been pointed out (McAven and Butler 1999), and this must necessarily be reproduced by either an algebraic expression or a combinatorial recipe.
- It may be possible to generate the representation matrices associated with the permutations in the standard basis of the symmetric group, since the permutation matrices transform between differently labelled tableaux (McAven and Butler 1999).
- The representation matrices are calculated in a number of ways, including a physics-based projected spin function approach (Sarma *et al* 1996, Rettrup and Pauncz 1996). The previous point may therefore mean we can provide useful insight into such approaches, and more particularly the physics behind them.

We have presented, in section 6, an example of how the search method is initially iterated. This is seen to be unsuccessful, in the sense that we do not find a consistent angular momentum description for the primitive one-box tableaux. The section nevertheless serves as a useful illustration of the method itself. It should be emphasized that our method is not an algorithm in the classical sense; despite following a step by step methodology the choices are generally from infinite sets.

Acknowledgment

One of the authors (LFM) appreciates support by The Foundation for Research, Science and Technology (New Zealand) contract number WND801.

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